Aristotelian and Duality Relations Beyond the Square of Opposition

Lorenz Demey[1] and Hans Smessaert[2]

[1] Center for Logic and Analytic Philosophy, KU Leuven
lorenz.demey@kuleuven.be
[2] Department of Linguistics, KU Leuven
hans.smessaert@kuleuven.be

Abstract. Nearly all squares of opposition found in the literature represent both the Aristotelian relations and the duality relations, and exhibit a very close correspondence between both types of logical relations. This paper investigates the interplay between Aristotelian and duality relations in diagrams beyond the square. In particular, we study a Buridan octagon, a Lenzen octagon, a Keynes-Johnson octagon and a Moretti octagon. Each of these octagons is a natural extension of the square, both from an Aristotelian perspective and from a duality perspective. The results of our comparative analysis turn out to be highly nuanced.

Keywords: Aristotelian relations, duality relations, square of opposition, Aristotelian diagram, duality diagram, logical geometry.

1 Introduction

The square of opposition represents four propositions, and certain logical relations holding between them. This diagram has a long and well-documented history in philosophy and logic [36]. In contemporary (analytic) philosophy, it has been used in various areas, such as philosophy of language, epistemology, philosophy of religion, ethics, and philosophy of law. In logic, the square of opposition has been used to study systems of modal logic, various non-classical logics, probabilistic and fuzzy logics, and logics of rational agency. Finally, because of the ubiquity of the logical relations that it represents, the square is nowadays also frequently used outside the boundaries of philosophy and logic, in disciplines such as psychology, linguistics and computer science. A comprehensive overview of this wide diversity of applications (including many bibliographic references) can be found in [15] and [16]. The square of opposition visually represents the *Aristotelian relations*: contradiction, contrariety, subcontrariety, and subalternation. However, most — nearly all — squares that appear in the literature also exhibit another type of logical relations, viz. the *duality relations*: internal negation, external negation and duality. Based on the concrete diagrams found in the literature, the notions of *Aristotelian square* and *duality square* thus seem to be almost co-extensional with each other. Nevertheless, there also seem to be clear conceptual differences between both types of logical diagrams.

The research program of *logical geometry* is concerned with the systematic study of logical diagrams in general, and Aristotelian diagrams and duality diagrams in particular. We investigate these diagrams using cognitive and geometric notions, such as informational vs. computational equivalence [12,14], Euclidean distance [16,44], vertex-first projections [10] and subdiagrams [6,42]. On the logical side, we focus on issues such as diagram informativity [43], logic-sensitivity [8], diagram classification [45] and Boolean structure [15,46]. The visual and logical properties of Aristotelian diagrams and duality diagrams are thus relatively well-understood in isolation. However, we do not yet have a clear picture of the precise interconnections between these two types of logical diagrams. Smessaert [41] has achieved some promising results in this direction, by moving beyond the square of opposition and focusing on a specific hexagon of opposition.

The main goal of this paper is to further advance this line of research, by analyzing the interplay between Aristotelian and duality relations in several octagons of opposition. These octagons will be shown to be very natural extensions/generalizations of the classical square of opposition, both from an Aristotelian perspective and from a duality perspective. With respect to the latter, we will discuss two main generalizations of 'classical' duality, viz. composed operator duality and generalized Post duality. This approach constitutes a major improvement over that of [41], since the comparative analysis there is based on a hexagon, which naturally extends the classical square from an Aristotelian perspective, but arguably *not* from a duality perspective. Consequently, the comparative analysis in this paper will provide a more solid basis for drawing conclusions regarding the interconnections between Aristotelian and duality diagrams. In particular, we will first focus on the individual Aristotelian and duality relations, and argue that the systematic correspondence, as found in the square of opposition, is lost in the octagons (albeit to varying degrees).[1] Furthermore, there is no systematic correspondence at the level of entire diagrams either. Nevertheless, we show that at a higher level of abstraction, the correspondence does seem to remain intact (again, to varying degrees).

The paper is organized as follows. Section 2 describes the interplay between Aristotelian and duality relations in the classical square of opposition. Next, Section 3 discusses the (in)dependence of these two types of relations, and examines Smessaert's [41] comparative analysis based on a hexagon of opposition. Sections 4 and 5 constitute the core of this paper. Section 4 is concerned with composed operator duality, and analyzes the interplay between Aristotelian and duality relations in a Buridan octagon and a Lenzen octagon. Next, Section 5 focuses on generalized Post duality, and analyzes the interplay between Aristotelian and duality relations in a Keynes-Johnson octagon and a Moretti octagon. Finally, Section 6 summarizes the results obtained in this paper, and mentions some questions for future research.

[1] Similar conclusions were reached in [41], but there, one could still object that the loss of correspondence beyond the square is merely due to the fact that from a duality perspective, the hexagon is not a natural generalization of the square of opposition. Such an objection cannot be raised against the conclusions drawn in this paper.

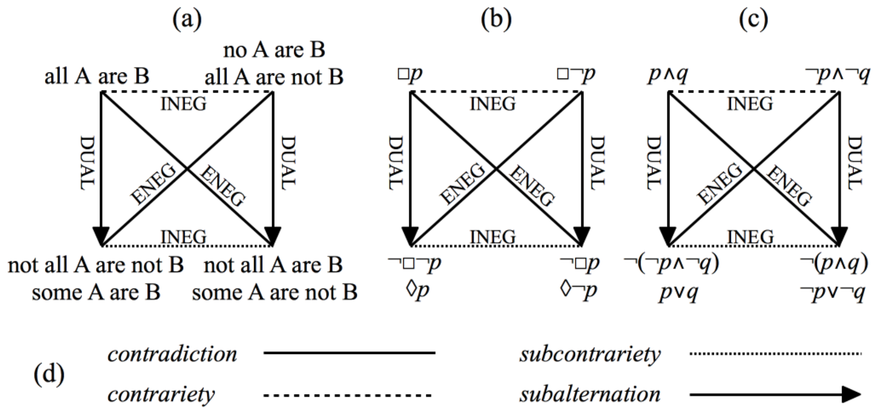

Fig. 1. Squares of opposition for (a) syllogistics, (b) modal logic, (c) propositional logic; (d) code for visualizing the Aristotelian relations.

2 Aristotelian and Duality Squares

We begin by considering the three most well-known squares of opposition. Without a doubt, the oldest and most frequently used square of opposition is that for *syllogistics*, as shown in Fig. 1(a). Both with respect to history and to frequency of use, a close second is the square of opposition for *modal logic*, as shown in Fig. 1(b). Furthermore, with the seminal work of authors such as Boole, De Morgan and Frege in the 19th and early 20th century also came the square of opposition for *propositional logic*, as shown in Fig. 1(c). Each of these square diagrams exhibits four key propositions of their underlying logical system, and the Aristotelian relations holding between them. These relations can be defined on various levels of generality and abstractness [13], but for our current purposes it will suffice to consider the most informal definition: two propositions are

contradictory	iff	they cannot be true together they cannot be false together,	and
contrary	iff	they cannot be true together they can be false together,	and
subcontrary	iff	they can be true together they cannot be false together,	and
in subalternation	iff	the first one entails the second one the second one does not entail the first one.	and

These relations will be abbreviated as *CD*, *C*, *SC* and *SA*, respectively, and visualized according to the code in Fig. 1(d). For example, in Fig. 1 we observe that CD(*some A are B*, *no A are B*) in the syllogistic square, $C(\Box p, \Box\neg p)$ and $SC(\Diamond p, \Diamond\neg p)$ in the modal square, and $SA(p\wedge q, p\vee q)$ in the propositional square.

The contradiction relation is the most important Aristotelian relation,[2] and accordingly, it plays a crucial role in Aristotelian diagrams. Each proposition φ has a unique contradictory (up to logical equivalence), viz. $\neg\varphi$. The square of opposition, and almost all other Aristotelian diagrams found in the literature as well, are closed under contradiction: if the diagram contains φ, then it also contains $\neg\varphi$.[3] The propositions occurring in an Aristotelian diagram can thus naturally be grouped into *pairs of contradictory propositions* (PCDs). Consequently, a square of opposition should not simply be seen as consisting of 4 'individual' propositions, but rather of 2 PCDs. This perspective also suggests a natural way of extending the square, viz. by adding more PCDs. We thus go from 2 PCDs to 3 PCDS, 4 PCDs, etc. — or in more geometric/diagrammatic terms: from square to hexagon, octagon, etc.[4]

The squares of opposition in Fig. 1(a–c) not only represent the Aristotelian relations, but also the duality relations. Just as before, these relations can be defined on various levels of generality and abstractness [13], but for our current purposes it will again suffice to consider the most informal definition. Suppose that φ and ψ are the results of applying n-ary operators O_φ and O_ψ to the same n propositions $\alpha_1, \ldots, \alpha_n$, i.e. $\varphi \equiv O_\varphi(\alpha_1, \ldots, \alpha_n)$ and $\psi \equiv O_\psi(\alpha_1, \ldots, \alpha_n)$. We then say that φ and ψ are each other's

$$\begin{array}{lll} \textit{external negation} & \text{iff} & O_\varphi(\alpha_1,\ldots,\alpha_n) \equiv \neg O_\psi(\alpha_1,\ldots,\alpha_n), \\ \textit{internal negation} & \text{iff} & O_\varphi(\alpha_1,\ldots,\alpha_n) \equiv O_\psi(\neg\alpha_1,\ldots,\neg\alpha_n), \\ \textit{dual} & \text{iff} & O_\varphi(\alpha_1,\ldots,\alpha_n) \equiv \neg O_\psi(\neg\alpha_1,\ldots,\neg\alpha_n). \end{array}$$

These relations will be abbreviated as ENEG, INEG and DUAL, respectively. Note that INEG operates on *all* propositions $\alpha_1, \ldots, \alpha_n$. In Fig. 1 we see that ENEG(*some A are B*, *no A are B*) in the syllogistic square, INEG($\Box p, \Box\neg p$) and INEG($\Diamond p, \Diamond\neg p$) in the modal square, and DUAL($p \wedge q, p \vee q$) in the propositional square.

The logical behavior of the duality relations is well-understood [11,45]. In particular, these relations are all functional (up to logical equivalence); for example, if INEG(φ, ψ_1) and INEG(φ, ψ_2), then $\psi_1 \equiv \psi_2$. Hence we can also view them as functions, and write, for example, $\psi =$ INEG(φ) instead of INEG(φ, ψ). Furthermore, since the INEG-relation is symmetrical — i.e. INEG(φ, ψ) iff INEG(ψ, φ) —, the INEG-function is idempotent: INEG(INEG(φ)) $= \varphi$. (All of this applies not only to INEG, but also to ENEG and DUAL.) In sum, the three duality functions, together with the identity function ID (defined by ID(φ) $:= \varphi$ for all φ) form a Klein 4-group under composition ($\circ$) [1,37], with the following Cayley table:

[2] Note that the definitions of contrariety and subcontrariety can both be seen as weakened versions of that of contradiction. It can also be shown that contradiction is the most informative of the Aristotelian relations [43].

[3] Furthermore, the contradiction relation is usually visualized by means of *central symmetry*, so that all pairs of contradictory propositions are represented by diagonals that intersect each other in the Aristotelian diagram's center of symmetry [10,12,14].

[4] In this paper we will not distinguish between different geometrical representations of the same set of PCDs. For example: (i) 3 PCDs can be visualized as a hexagon or as an octahedron; (ii) 4 PCDs can be visualized as an octagon or as a cube [12,14].

$\circ$	ID	ENEG	INEG	DUAL
ID	ID	ENEG	INEG	DUAL
ENEG	ENEG	ID	DUAL	INEG
INEG	INEG	DUAL	ID	ENEG
DUAL	DUAL	INEG	ENEG	ID

The Klein 4-group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. This group-theoretical isomorphism gives us a firm syntactic handle on the duality relations: each copy of $\mathbb{Z}_2$ governs an independent negation position: the first copy corresponds to external negation, and the second corresponds to internal negation.[5] This also suggests a natural way of extending duality behavior beyond the square of opposition (i.e. beyond the Klein 4-group), viz. by adding more independent negation positions (i.e. by adding more copies of $\mathbb{Z}_2$). We thus go from $\mathbb{Z}_2 \times \mathbb{Z}_2$ (2 negation positions, yielding a group of $2^2 = 4$ duality functions) to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (3 negation positions, yielding a group of $2^3 = 8$ duality functions), etc.

If we now bring the Aristotelian and duality perspectives together, we see that the squares of opposition in Fig. 1(a–c) exhibit a highly uniform correspondence between both types of logical relations. In particular, there is a correspondence between (i) the Aristotelian relation *CD* and the duality relation ENEG, (ii) the Aristotelian relations *C* and *SC* and the duality relation INEG, and (iii) the Aristotelian relation *SA* and the duality relation DUAL. Each square thus gives rise to an Aristotelian/duality multigraph (ADM) as shown in Fig. 2. This ADM visualizes, for each combination of an Aristotelian and a duality relation, how many times that specific combination occurs in the square of opposition.[6] Although the correspondence between Aristotelian and duality relations is not perfect, the ADM clearly shows that it is still highly regular. Using graph-theoretical terminology [18], the ADM for the square of opposition has 4 *connected components*, viz. $\{CD, \text{ENEG}\}$, $\{C, SC, \text{INEG}\}$, $\{SA, \text{DUAL}\}$ and $\{EQ, \text{ID}\}$. More concretely, each Aristotelian relation corresponds to a unique duality relation; vice versa, the duality relations ENEG, DUAL and ID correspond to a unique Aristotelian relation, while INEG corresponds to two Aristotelian relations.

In summary, the well-known squares of opposition from Fig. 1(a–c) show that there is a clear correspondence between Aristotelian and duality considerations. At the level of *diagrams*, these squares of opposition are simultaneously Aris-

[5] Under the group-theoretical isomorphism between the Klein 4-group for the duality functions and $\mathbb{Z}_2 \times \mathbb{Z}_2$, ID corresponds to $(0,0)$ (apply no negations at all), ENEG to $(1,0)$ (only apply external negation), INEG to $(0,1)$ (only apply internal negation) and DUAL to $(1,1)$ (apply both external and internal negation). If the duality function f corresponds to $(i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, we thus get $f(O(\alpha_1, \ldots, \alpha_n)) = \neg^i O(\neg^j \alpha_1, \ldots, \neg^j \alpha_n)$ (with the usual definitions $\neg^0 \varphi := \varphi$ and $\neg^1 \varphi := \neg\varphi$).

[6] Note that the ADM includes *EQ* (logical equivalence) as the Aristotelian counterpart of ID. Strictly speaking, *EQ* is not one of the Aristotelian relations, but it is closely related to them [43], and it is implicitly present whenever we write multiple, logically equivalent propositions in a single vertex of an Aristotelian diagram. (Each vertex thus has an *EQ*-loop to itself.) Note, in this context, that the square of opposition is sometimes also referred to as 'the square of opposition *and equipollence*' [33].

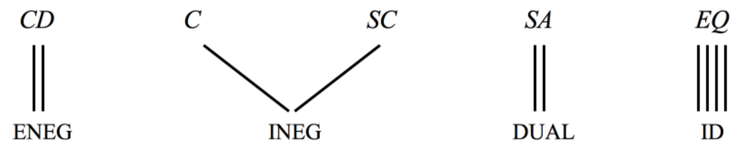

Fig. 2. Aristotelian/duality multigraph (ADM) for the classical square of opposition.

totelian squares and duality squares. At the level of the individual *relations*, the correspondence is summarized by the ADM in Fig. 2. Finally, it bears emphasizing that this correspondence can also be observed in more recent (and thus lesser-known) squares of opposition, such as those for public announcement logic [7], future contingents [21], definite descriptions [9], and rough set theory [48].

3 (In)dependence of Aristotelian and Duality Diagrams

Because of this correspondence, several authors [5,39,48] come close to outright identifying the two types of squares — e.g. by using Aristotelian terminology to describe the duality square (or vice versa), or by viewing one as a generalization of the other. The correspondence was already noted in medieval logic: influential authors such as Peter of Spain [4], William of Sherwood [29] and John Wyclif [19] discussed the mnemonic rhyme *pre contradic, post contra, pre postque subalter*, in which external negation (*pre*) is associated with contradiction, internal negation (*post*) with contrariety, and duality (*pre postque*) with subalternation.[7]

Despite this close correspondence, there are still some crucial differences between Aristotelian and duality diagrams [3,47]. Regarding the individual relations, it should be pointed out that (i) the duality relations are all symmetric, whereas the Aristotelian relation SA is asymmetric, and that (ii) the duality relations are all functional, whereas the Aristotelian relations C, SC and SA are not (i.e. a single proposition can have multiple, non-equivalent contraries, subcontraries, and subalterns). Furthermore, the Aristotelian relations are far more sensitive to the details of the underlying logical system than the duality relations [8]. Consider, for example, the propositions $\Box p$ and $\Box\neg p$. In the normal modal logic KD, these two propositions are contrary to each other, but in the weakest normal modal logic, K, they do not stand in any Aristotelian relation at all [23]. Nevertheless, in both KD and K, these two propositions are each other's internal negation. In general, as long as two logical systems have classical Boolean

[7] This rhyme is incomplete, because as we have seen above (Fig. 2), internal negation (*post*) should not just be associated with contrariety, but also with subcontrariety [29, Footnote 54]. However, this omission can be explained in terms of the famous non-lexicalization of the O-corner [22]. The fact that *no A are B* is the internal negation of *all A are B* (i.e. *no* $\equiv$ *all* $\neg$, or in Latin: *nullus* $\equiv$ *omnis* $\neg$) is a contingent, empirical fact about English (resp. Latin), and should thus be captured by the rhyme. By contrast, the fact that *some A are not B* is the internal negation of *some A are B* (i.e. *some not* $\equiv$ *some* $\neg$, or in Latin: *aliquis non* $\equiv$ *aliquis* $\neg$) is almost analytically true, and thus need not be captured by the mnemotechnic rhyme.

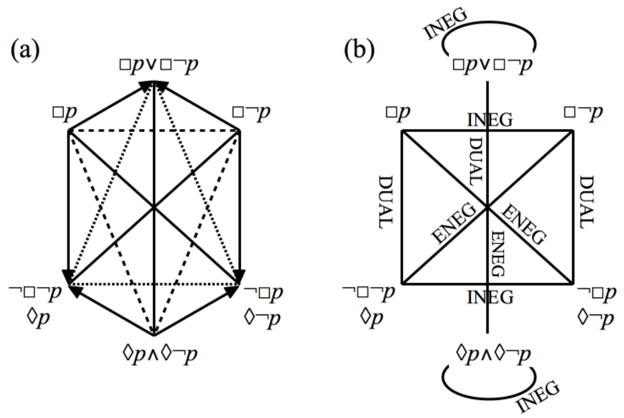

Fig. 3. (a) Aristotelian relations and (b) duality relations in the modal JSB hexagon.

connectives, they will yield the same duality relations, even though they might yield vast differences in the Aristotelian relations.

Perhaps the most powerful way to argue for the independence of Aristotelian and duality diagrams consists in analyzing diagrams *beyond* the square. For example, Smessaert [41] has studied the interplay between Aristotelian and duality relations in a *hexagon* of opposition, as shown in Fig. 3.[8] From an Aristotelian perspective, this hexagon is a very natural extension of the square: it is obtained by adding one pair of contradictory propositions (PCD), thus moving from a diagram with 2 PCDs to one with 3 PCDs. This type of hexagon is very well-known [15,42]; it was first studied in the 1950s by Jacoby [25], Sesmat [40] and Blanché [2], and is therefore called a 'Jacoby-Sesmat-Blanché (JSB) hexagon'.

This hexagon clearly illustrates the discrepancy in functionality between the Aristotelian and the duality relations. For example, $\Box p$ has a unique internal negation, viz. $\Box\neg p$, but it has multiple contraries, e.g. $\Box\neg p$ and $\Diamond p \wedge \Diamond\neg p$. The contrariety between $\Box p$ and $\Box\neg p$ thus corresponds to an INEG-relation (just like in the square), but the contrariety between $\Box p$ and $\Diamond p \wedge \Diamond\neg p$ does not correspond to any duality relation at all (which we will denote as $\varnothing$). Similarly, $\Box p$ has a unique dual, viz. $\Diamond p$, but it has multiple subalterns, e.g. $\Diamond p$ and $\Box p \vee \Box\neg p$. The subalternation from $\Box p$ to $\Diamond p$ thus corresponds to a DUAL-relation (just like in the square), but the subalternation between $\Box p$ to $\Box p \vee \Box\neg p$ does not correspond to any duality relation at all ($\varnothing$). Furthermore, $\Diamond p \wedge \Diamond\neg p$ turns out to be its own internal negation, so this INEG-relation will correspond to a logical equivalence (EQ). Consequently, $\Box p \vee \Box\neg p$ is not only the external negation of $\Diamond p \wedge \Diamond\neg p$, but also its dual; these ENEG- and DUAL-relations thus both correspond to the Aristotelian CD-relation. The entire configuration of Aristotelian and duality relations in the modal JSB hexagon is summarized by the ADM in Fig. 4.

[8] For reasons of space, we only consider the *modal* hexagon, which extends the modal square in Fig. 1(b). In exactly the same way, one could also extend the other two squares in Fig. 1(a)/(c) to hexagons, and draw the same conclusions about them.

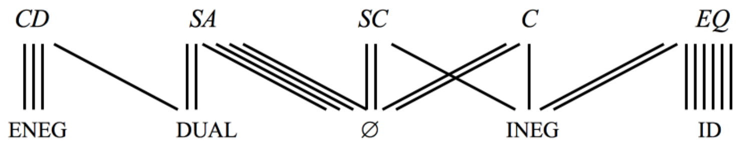

Fig. 4. Aristotelian/duality multigraph (ADM) for the modal JSB hexagon.

By comparing the ADM for the square (Fig. 2) with that for the JSB hexagon (Fig. 4), we immediately see that the latter is much more 'cluttered'. Instead of having 4 connected components, the entire multigraph in Fig. 4 is connected. Each Aristotelian relation corresponds to multiple duality relations (or the complete absence of any duality relation, $\varnothing$); vice versa, DUAL corresponds to two Aristotelian relations, while INEG corresponds to two Aristotelian relations and logical equivalence (EQ). In sum: the systematic correspondence between Aristotelian and duality relations is completely lost in the JSB hexagon.

One might object to the conclusion of this analysis. After all, the JSB hexagon is a natural extension of the square from an Aristotelian perspective, but *not* from a duality perspective. The hexagon is obtained by adding an extra PCD, but in terms of duality, this does not correspond to adding an extra negation position. Consequently, the hexagon cannot be seen as a single, 'unified' duality diagram, but should rather be seen as the superposition of two separate, independent duality diagrams, viz. the original duality square and the extra PCD (which are classified in [45] as a CLCL1 duality square and a collapsed, self-internal duality square, respectively). The independence of these two duality diagrams is illustrated by the high number of edges involving $\varnothing$ in the ADM in Fig. 4.

In the remainder of the paper, we will thus consider diagrams that are natural extensions of the square of opposition from both an Aristotelian and a duality perspective. In particular, we focus on *octagons of opposition*: these extend the Aristotelian square (from 2 PCDs to 4 PCDs, i.e. from $2 \times 2 = 4$ to $4 \times 2 = 8$ propositions) as well as the duality square (from 2 negation positions to 3 negation positions, i.e. from $2^2 = 4$ to $2^3 = 8$ propositions). We thus consider a new duality group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, which has been studied purely abstractly [35], but which also has two distinct concrete interpretations, viz. *composed operator duality* and *generalized Post duality*. These two types of duality, and the octagons that they give rise to, will be discussed in Sections 4 and 5, respectively.

4 Octagons for Composed Operator Duality

Suppose that φ is the result of applying an n-ary composed operator $O_1 \circ O_2$ to n propositions $\alpha_1, \ldots, \alpha_n$, i.e. $\varphi \equiv (O_1 \circ O_2)(\alpha_1, \ldots, \alpha_n) = O_1(O_2(\alpha_1, \ldots, \alpha_n))$.[9] For example, in modal logic we can view $\Box(p \wedge q)$ as the result of applying the composed operator $\Box \circ \wedge$ to the propositions p and q. (Westerståhl [47] discusses

[9] If O_2 is n-ary, the composed operator $O_1 \circ O_2$ will also be n-ary. Furthermore, O_1 will be assumed to be unary, but this assumption is not essential.

a linguistic example, viz. possessives with quantifiers; e.g. *some athletes of each country.*) By moving to composed operators, we have added an extra negation position, viz. *intermediate negation.* The proposition $O_1(O_2(\alpha_1, \ldots, \alpha_n))$ has

- a unique external negation (ENEG): $\neg O_1(\ O_2(\ \alpha_1, \ldots,\ \alpha_n)),$
- a unique intermediate negation (MNEG): $O_1(\neg O_2(\ \alpha_1, \ldots,\ \alpha_n)),$
- a unique internal negation (INEG): $O_1(\ O_2(\neg\alpha_1, \ldots, \neg\alpha_n)).$

Since each of these 3 independent negation positions may or may not be occupied, $O_1 \circ O_2$ gives rise to $2^3 = 8$ propositions in total, which exhibit a much richer duality behavior [6]. We now have three negation operations, and thus three pairwise combinations: ENEG∘INEG, ENEG∘MNEG, and MNEG∘INEG (abbreviated as EI, EM and MI, respectively). Finally, there is ENEG∘MNEG∘INEG (abbreviated as EMI), which operates on all three negation positions simultaneously.

4.1 The Buridan Octagon in Modal Syllogistics

In the logical works of the medieval philosopher John Buridan, we find three distinct octagons that exhibit composed operator duality [17,28,38]. We will focus on (a simplified version of) Buridan's modal octagon, which contains quantified *de re* modal propositions such as $\forall x \Box Px$. This proposition is the result of applying the composed operator $\forall \circ \Box$ to Px. This octagon can be thought of as capturing the interaction between the syllogistic square and the modal square from Fig. 1(a–b), and is thus a natural extension of both of these squares [17]. The logical behavior of this type of diagrams is well-studied; within the classification of Aristotelian diagrams, it is called a 'Buridan octagon' (for obvious historical reasons) [15].

The modal octagon is shown in Fig. 5(a).[10] For example, we observe that $\forall x \Box Px$ is contrary to three propositions, viz. (i) $\forall x \Box \neg Px$, (ii) $\forall x \neg \Box Px$, and (iii) $\neg \forall x \neg \Box \neg Px$. The first of these contrarieties corresponds to an INEG-relation, the second one to an MNEG-relation, and the third one to an EMI-relation. There are also four pairs of propositions that do not stand in any Aristotelian relation at all; Buridan himself called these *disparatae*; today, such pairs are called *unconnected* (*Un*) [43]. Two *Un*-pairs correspond to EI-relations, while the two others correspond to INEG-relations. The entire distribution of Aristotelian/duality relations in Buridan's modal octagon is summarized by the ADM in Fig. 6.

This ADM has only 3 connected components. Two of these components are $\{CD, \text{ENEG}\}$ and $\{EQ, \text{ID}\}$, which represent two clear-cut correspondences between Aristotelian and duality relations. However, in all other cases, the correspondence is highly irregular. Apart from *CD*, all Aristotelian relations correspond to multiple duality relations. Vice versa, MI and EM correspond to a unique Aristotelian relation (just like ENEG), but all remaining duality relations correspond to multiple Aristotelian relations. All of this illustrates the lack of any

[10] To avoid cluttering the diagrams, we will henceforth not explicitly show the *CD*- and ENEG-relations. These occur exactly at the diagram's diagonals, which intersect each other in the diagram's center of symmetry (recall Footnote 3).

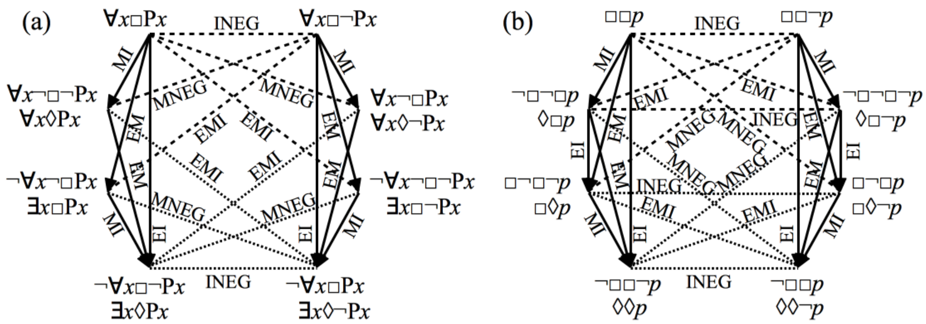

Fig. 5. (a) Buridan octagon in modal syllogistics; (b) Lenzen octagon in S4.2.

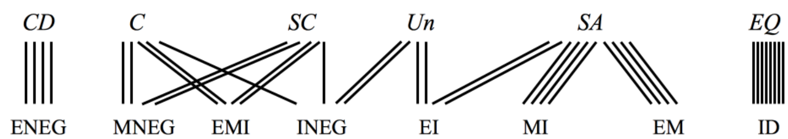

Fig. 6. ADM for the Buridan octagon in modal syllogistics.

systematic correspondence between Aristotelian and duality relations in Buridan's modal octagon. Furthermore, it should also be emphasized that this lack of correspondence cannot be due to Buridan's octagon purportedly not being a natural extension of the square of opposition from a duality perspective (unlike the JSB hexagon that was analyzed in Section 3). After all, we have already seen above that this octagon is a natural extension of the square from both an Aristotelian and a duality perspective.[11]

4.2 The Lenzen Octagon in S4.2

Another example of composed operator duality can be observed in the octagon in Fig. 5(b). This octagon is a natural extension of the modal square in Fig. 1(b): it is based on 'doubly modalized' propositions such as $\Box\Box p$, which can be seen as the result of applying the composed operator $\Box \circ \Box$ to the proposition p. In the well-known normal modal logic S4.2, these propositions stand in the Aristotelian relations shown in Fig. 5(b). (The key axiom of S4.2 is $\Diamond\Box p \rightarrow \Box\Diamond p$ [23].) Furthermore, some of these propositions can be simplified to 'singly modalized' propositions (e.g. $\Box\Box p$ is logically equivalent to $\Box p$ in S4.2), but we have not done so, in order to emphasize the composed operator duality exhibited by this octagon. This octagon belongs to a well-known type of Aristotelian diagrams, viz. the 'Lenzen octagons' (which is so-called because a diagram of this type was first used by Lenzen [30]). A Lenzen octagon has recently also been used in [9].

Looking at the octagon in Fig. 5(b), we observe, for example, that $\neg\Box\Box p$ is subcontrary to three propositions, viz. (i) $\neg\Box\Box\neg p$, (ii) $\Box\neg\Box\neg p$, and (iii) $\neg\Box\neg\Box p$.

[11] Compare the ADMs for the modal JSB hexagon and Buridan's modal octagon in Figs. 4 and 6, and note the absence of $\varnothing$ in the latter.

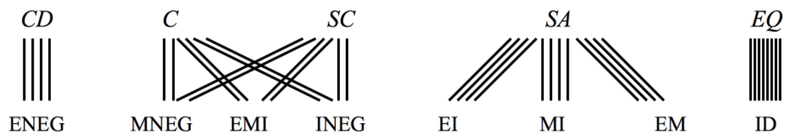

Fig. 7. ADM for the Lenzen octagon in S4.2.

The first of these subcontrarieties corresponds to an INEG-relation, the second one to an EMI-relation, and the third one to an MNEG-relation. We also note that EMI($\Box\Box p, \neg\Box\neg\Box\neg p$) and EMI($\Box\neg\Box\neg p, \neg\Box\Box p$); the first of these EMI-relations corresponds to a contrariety, while the second one corresponds to a subcontrariety. The entire distribution of Aristotelian and duality relations in the Lenzen octagon in S4.2 is summarized by the ADM in Fig. 7.

This ADM shows that the correspondence between Aristotelian and duality relations in the Lenzen octagon in S4.2 is again quite irregular, although not as bad as in Buridan's modal octagon (recall Fig. 6). Apart from *CD*, all Aristotelian relations again correspond to multiple duality relations. Vice versa, however, only MNEG, EMI and INEG correspond to multiple Aristotelian relations — all other duality relations correspond to a unique Aristotelian relation.

When we compare the ADM for the Lenzen octagon (cf. Fig. 7) with that for the square of opposition (cf. Fig. 2), the similarities between both ADMs seem to prevail, rather than the dissimilarities. Both ADMs have 4 connected components, two of which are $\{CD, \text{ENEG}\}$ and $\{EQ, \text{ID}\}$, which represent clear-cut correspondences between Aristotelian and duality relations. Furthermore, the component $\{C, SC, \text{INEG}\}$ from the square has expanded into $\{C, SC, \text{MNEG}, \text{EMI}, \text{INEG}\}$. The composed operator duality relations MNEG, EMI and INEG thus jointly fulfill the role of the original INEG-relation, in corresponding to *C* and *SC*. Similarly, the component $\{SA, \text{DUAL}\}$ from the square has expanded into $\{SA, \text{MI}, \text{EM}, \text{EI}\}$, i.e. the composed operator duality relations MI, EM and EI jointly fulfill the role of the original DUAL-relation, in corresponding to *SA*.

5 Octagons for Generalized Post Duality

Recall that with classical duality, we assume that internal negation is applied to *all* argument positions, i.e. if O is an n-ary operator, the internal negation of $O(\alpha_1, \ldots, \alpha_n)$ is defined as $O(\neg\alpha_1, \ldots, \neg\alpha_n)$ (also cf. Footnote 5). However, we can also drop this assumption, and let internal negation apply to each argument position independently [24,31]. In the case of a binary operator O, we thus have 3 independent negation positions in total:[12] the proposition $O(\alpha_1, \alpha_2)$ has

- a unique external negation (ENEG): $\neg O(\alpha_1, \alpha_2)$,
- a unique first internal negation (INEG1): $O(\neg\alpha_1, \alpha_2)$,
- a unique second internal negation (INEG2): $O(\alpha_1, \neg\alpha_2)$.

[12] In general, for an n-ary operator, we have $n + 1$ independent negation positions, viz. 1 external negation and n internal negations (one for each argument position).

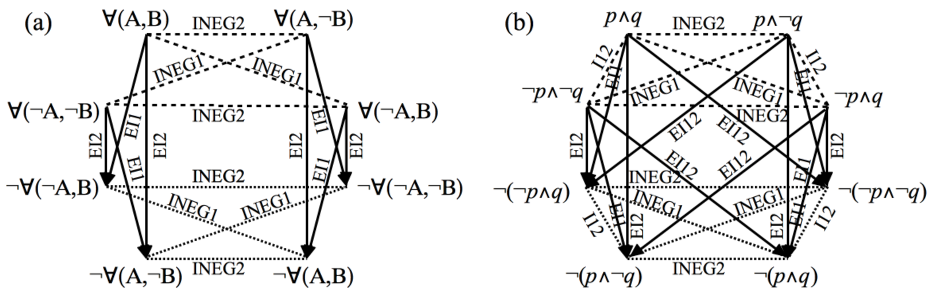

Fig. 8. (a) Keynes-Johnson octagon in syllogistics with subject negation — note that $\forall(A, B)$ should be read as *all A are B*); (b) Moretti octagon in propositional logic.

Since each of these 3 independent negation positions may or may not be occupied, we obtain $2^3 = 8$ propositions in total, which again exhibit a much richer duality behavior. We now have three negation operations, and thus three pairwise combinations: ENEG $\circ$ INEG1, ENEG $\circ$ INEG2, and INEG1 $\circ$ INEG2 (abbreviated as EI1, EI2 and I12, respectively). Finally, there is ENEG $\circ$ INEG1 $\circ$ INEG2 (abbreviated as EI12), which operates on all three negation positions simultaneously.

5.1 The Keynes-Johnson Octagon in Syllogistics with Subject Negation

Classically, a categorical statement of the form *all A are B* is seen as the result of applying the unary operator *all A* to the predicate B — which gives rise to the square of opposition in Fig. 1(a). However, we can also view such a statement as the result of applying the binary operator *all* to the predicates A and B. If these two predicates can be negated independently, we obtain 8 propositions in total. Assuming that the extensions of A and B are neither empty nor the entire universe of discourse, these 8 propositions constitute the octagon of opposition shown in Fig. 8(a), which was first studied by Keynes [27] and Johnson [26]. The logical behavior of this Aristotelian diagram is well-studied [8,20]; from a classificatory perspective, it is called a 'Keynes-Johnson octagon' [13].

Looking at the octagon in Fig. 8(a), we observe, for example, that $\forall(A, B)$ is contrary to two propositions, viz. $\forall(A, \neg B)$ and $\forall(\neg A, B)$. The first of these contrarieties corresponds to an INEG2-relation, and the second one to an INEG1-relation. We also note that the INEG2 of $\forall(A, B)$ is $\forall(A, \neg B)$ and that the INEG2 of $\neg\forall(A, \neg B)$ is $\neg\forall(A, B)$; the first of these INEG2-relations corresponds to a contrariety, and the second to a subcontrariety. The entire distribution of Aristotelian and duality relations in the Keynes-Johnson octagon for syllogistics with subject negation is summarized by the ADM in Fig. 9.

This ADM shows that the correspondence between Aristotelian and duality relations in the Keynes-Johnson octagon is quite regular. Apart from *CD*, all Aristotelian relations correspond to multiple duality relations. Vice versa, however, only INEG1 and INEG2 correspond to multiple Aristotelian relations

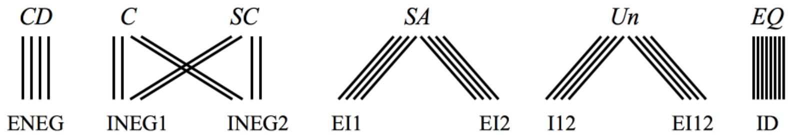

Fig. 9. ADM for the Keynes-Johnson octagon in syllogistics with subject negation.

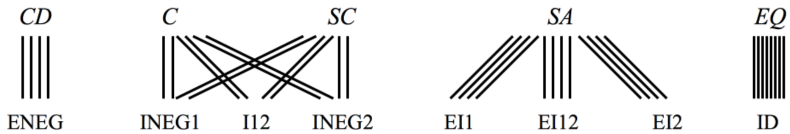

Fig. 10. ADM for the Moretti octagon in propositional logic.

— all other duality relations correspond to a unique Aristotelian relation. The ADM has 5 connected components, two of which are $\{CD, \text{ENEG}\}$ and $\{EQ, \text{ID}\}$, which represent clear-cut correspondences between Aristotelian and duality relations. In comparison with the ADM for the square (cf. Fig. 2), the component $\{SA, \text{DUAL}\}$ from the square has expanded into $\{SA, \text{EI}1, \text{EI}2\}$. The generalized Post duality relations EI1 and EI2 thus jointly fulfill the role of the original DUAL-relation, in corresponding to SA. Similarly, the component $\{C, SC, \text{INEG}\}$ has expanded into $\{C, SC, \text{INEG}1, \text{INEG}2\}$, i.e. the generalized Post duality relations INEG1 and INEG2 thus jointly fulfill the role of the original INEG-relation, in corresponding to C and SC.

5.2 The Moretti Octagon in Propositional Logic

Another example of generalized Post duality can be observed in the octagon in Fig. 8(b). This octagon is a natural extension of the propositional logic square in Fig. 1(c). It was first studied by Moretti [34] and later also by others [32]. Within the classification of Aristotelian diagrams, it is called a 'Moretti octagon'.

Looking at the octagon in Fig. 8(b), we observe, for example, that $p \wedge q$ is contrary to three propositions, viz. $\neg p \wedge q$, $p \wedge \neg q$ and $\neg p \wedge \neg q$. The first of these contrarieties corresponds to an INEG1-relation, the second one to an INEG2-relation, and the third one to an I12-relation. We also note that the INEG1 of $p \wedge q$ is $\neg p \wedge q$ and that the INEG1 of $\neg(\neg p \wedge q)$ is $\neg(p \wedge q)$; the first of these INEG1-relations corresponds to a contrariety, and the second to a subcontrariety. The entire distribution of Aristotelian and duality relations in the Moretti octagon for propositonal logic is summarized by the ADM in Fig. 10.

This ADM shows that the correspondence between Aristotelian and duality relations in the Moretti octagon is quite regular. Apart from CD and EQ, all Aristotelian relations correspond to multiple duality relations. Vice versa, however, only INEG1, INEG2 and I12 correspond to multiple Aristotelian relations — all other generalized Post duality relations correspond to a unique Aristotelian relation. The ADM has 4 connected components, two of which are $\{CD, \text{ENEG}\}$ and $\{EQ, \text{ID}\}$, which represent clear-cut correspondences between Aristotelian and duality relations. In comparison with the ADM for the square (cf. Fig. 2), the component $\{SA, \textsc{dual}\}$ from the square has expanded into $\{SA, \textsc{ei}1, \textsc{ei}2, \textsc{ei}12\}$. The generalized Post duality relations EI1, EI2 and EI12 thus jointly fulfill the role of the original DUAL-relation, in corresponding to SA. Similarly, the component $\{C, SC, \textsc{ineg}\}$ from the square has expanded into $\{C, SC, \textsc{ineg}1, \textsc{ineg}2, \textsc{i}12\}$, i.e. INEG1, INEG2 and I12 thus jointly fulfill the role of the original INEG-relation, in corresponding to C and SC.

6 Conclusion

In this paper we have analyzed the correspondence between Aristotelian and duality relations in four octagons of oppositions. These octagons are all natural extensions of the square of opposition from both an Aristotelian and a duality perspective, and hence, they provide a solid basis for our comparative analysis. The results we obtained are quite nuanced.

On the one hand, the clear-cut correspondence between Aristotelian and duality relations that is found in many squares of opposition (cf. Fig. 2) is lost. In each octagon, we find several cases of a single Aristotelian relation corresponding to multiple duality relations, and vice versa (cf. Figs. 6, 7, 9 and 10). Furthermore, there is no uniform correspondence at the level of *diagrams* either: composed operator duality corresponds to (at least) two types of Aristotelian diagrams (viz. a Buridan octagon and a Lenzen octagon), and generalized Post duality also corresponds to (at least) two types of Aristotelian diagrams (viz. a Keynes-Johnson octagon and a Moretti octagon).

On the other hand, at a higher level of abstraction, the correspondence seems to remain largely intact. Recall that the ADM of the square has 4 connected components. In the ADMs of the Lenzen, Keynes-Johnson, and Moretti octagons, the number of connected components does not decrease. Furthermore, the connected components remain logically meaningful. For example, in the square, SA corresponds to DUAL, but in the Lenzen octagon, this Aristotelian relation corresponds to EI, MI and EM, in the Keynes-Johnson octagon to EI1 and EI2, and in the Moretti octagon to EI1, EI2 and EI12. Finally, note that the ADMs for the Lenzen and Moretti octagons (Figs. 7 and 10) are *isomorphic* to each other.

In future work, we will further investigate the correspondence between Aristotelian and duality diagrams. The results obtained in this paper will provide valuable input for such an investigation. Another research question is of a more historical nature. Apart from the square of opposition, the two oldest Aristotelian diagrams that have ever been used, are probably the Buridan octagon (14th century) and the Keynes-Johnson octagon (end of the 19th century). By contrast, the JSB hexagon — which is the most natural extension of the square from a strictly Aristotelian perspective — was only proposed in the 1950s. In this paper, we have argued that, unlike the JSB hexagon, the Buridan octagon and the Keynes-Johnson octagon can also be seen as duality diagrams (according to a suitably generalized notion of duality). Consequently, one might wonder whether these historical facts should primarily be explained in terms of the octagons' duality relations, rather than their Aristotelian relations.

Acknowledgements We would like to thank Margaux Smets and three anonymous reviewers for their valuable feedback.The first author holds a Postdoctoral Fellowship of the Research Foundation–Flanders (FWO).

References

1. van Benthem, J.: Linguistic universals in logical semantics. In: Zaefferer, D. (ed.) Semantic Universals and Universal Semantics, pp. 17–36. Foris (1991)
2. Blanché, R.: Sur l'opposition des concepts. Theoria 19, 89–130 (1953)
3. Chow, K.F.: General patterns of opposition squares and 2n-gons. In: Béziau, J.Y., Jacquette, D. (eds.) Around and Beyond the Square, pp. 263–275. Springer (2012)
4. Copenhaver, B.P., Normore, C.G., Parsons, T. (eds.): Peter of Spain, Summaries of Logic. Text, Translation, Introduction and Notes. Oxford University Press (2014)
5. D'Alfonso, D.: The square of opposition and generalized quantifiers. In: Béziau, J.Y., Payette, G. (eds.) Around and Beyond the Square of Opposition, pp. 219–227. Springer (2012)
6. Demey, L.: Algebraic aspects of duality diagrams. In: Cox, P.T., Plimmer, B., Rodgers, P. (eds.) Diagrams 2012, pp. 300–302. LNCS 7352, Springer (2012)
7. Demey, L.: Structures of oppositions for public announcement logic. In: Béziau, J.Y., Jacquette, D. (eds.) Around and Beyond the Square of Opposition, pp. 313–339. Springer (2012)
8. Demey, L.: Interactively illustrating the context-sensitivity of Aristotelian diagrams. In: Christiansen, H., Stojanovic, I., Papadopoulos, G. (eds.) Modeling and Using Context, pp. 331–345. LNCS 9405, Springer (2015)
9. Demey, L.: The logical geometry of Russell's theory of definite descriptions. Unpublished manuscript (2017)
10. Demey, L., Smessaert, H.: The relationship between Aristotelian and Hasse diagrams. In: Dwyer, T., Purchase, H., Delaney, A. (eds.) Diagrams 2014, pp. 213–227. LNCS 8578, Springer (2014)
11. Demey, L., Smessaert, H.: Duality in logic and language. In: Fieser, J., Dowden, B. (eds.) Internet Encyclopedia of Philosophy. University of Tennessee (2016)
12. Demey, L., Smessaert, H.: The interaction between logic and geometry in Aristotelian diagrams. In: Jamnik, M., Uesaka, Y., Schwartz, S.E. (eds.) Diagrams 2016, pp. 67–82. LNCS 9781, Springer (2016)
13. Demey, L., Smessaert, H.: Metalogical decorations of logical diagrams. Logica Universalis 10, 233–292 (2016)
14. Demey, L., Smessaert, H.: Shape heuristics in Aristotelian diagrams. In: Kutz, O., Borgo, S., Bhatt, M. (eds.) Shapes 3.0, pp. 35–45. CEUR-WS 1616 (2016)
15. Demey, L., Smessaert, H.: Combinatorial bitstring semantics for arbitrary logical fragments. Journal of Philosophical Logic DOI: 10.1007/s10992-017-9430-5 (2017)
16. Demey, L., Smessaert, H.: Logical and geometrical distance in polyhedral Aristotelian diagrams in knowledge representation. Symmetry 9 (2017)
17. Demey, L., Steinkrüger, P.: De logische geometrie van Johannes Buridanus' modale achthoek. Tijdschrift voor Filosofie 79, 217–238 (2017)
18. Diestel, R.: Graph Theory. Springer (2006)
19. Dziewicki, M.H. (ed.): Johannis Wyclif, Tractatus de Logica, Vol. 1. Trübner (1893)
20. Hacker, E.A.: The octagon of opposition. Notre Dame Journal of Formal Logic 16, 352–353 (1975)
21. Hess, E.: The open future square of opposition: A defense. Sophia (2017)

22. Horn, L.R.: A Natural History of Negation. University of Chicago Press (1989)
23. Hughes, G.E., Cresswell, M.J.: A New Introduction to Modal Logic. Routledge (1996)
24. Humberstone, L.: The Connectives. MIT Press (2011)
25. Jacoby, P.: A triangle of opposites for types of propositions in Aristotelian logic. New Scholasticism 24, 32–56 (1950)
26. Johnson, W.: Logic. Part I. Cambridge University Press (1921)
27. Keynes, J.N.: Studies and Exercises in Formal Logic. MacMillan (1884)
28. Klima, G. (ed.): John Buridan, Summulae de Dialectica. Yale University Press (2001)
29. Kretzmann, N.: William of Sherwood's Introduction to Logic. Minnesota Archive Editions (1966)
30. Lenzen, W.: How to square knowledge and belief. In: Béziau, J.Y., Jacquette, D. (eds.) Around and Beyond the Square of Opposition, pp. 305–311. Springer (2012)
31. Libert, T.: Hypercubes of duality. In: Béziau, J.Y., Jacquette, D. (eds.) Around and Beyond the Square of Opposition, pp. 293–301. Springer (2012)
32. Luzeaux, D., Sallantin, J., Dartnell, C.: Logical extensions of Aristotle's square. Logica Universalis 2, 167–187 (2008)
33. Mikhail, J.: Elements of Moral Cognition. Cambridge University Press (2011)
34. Moretti, A.: The Geometry of Logical Opposition. Ph.D. thesis, Neuchâtel (2009)
35. Moretti, A.: Why the logical hexagon? Logica Universalis 6, 69–107 (2012)
36. Parsons, T.: The traditional square of opposition. In: Zalta, E.N. (ed.) Stanford Encyclopedia of Philosophy (Summer 2017 Edition). CSLI (2017)
37. Peters, S., Westerståhl, D.: Quantifiers in Language and Logic. Oxford University Press (2006)
38. Read, S.: John Buridan's theory of consequence and his octagons of opposition. In: Béziau, J.Y., Jacquette, D. (eds.) Around and Beyond the Square of Opposition, pp. 93–110. Springer (2012)
39. Schumann, A.: On two squares of opposition: the Leśniewski's style formalization of synthetic propositions. Acta Analytica 28, 71–93 (2013)
40. Sesmat, A.: Logique II. Les Raisonnements. La syllogistique. Hermann (1951)
41. Smessaert, H.: The classical Aristotelian hexagon versus the modern duality hexagon. Logica Universalis 6, 171–199 (2012)
42. Smessaert, H., Demey, L.: Logical and geometrical complementarities between Aristotelian diagrams. In: Dwyer, T., Purchase, H., Delaney, A. (eds.) Diagrams 2014, pp. 246–260. LNCS 8578, Springer (2014)
43. Smessaert, H., Demey, L.: Logical geometries and information in the square of opposition. Journal of Logic, Language and Information 23, 527–565 (2014)
44. Smessaert, H., Demey, L.: Visualising the Boolean algebra $\mathbb{B}_4$ in 3D. In: Jamnik, M., Uesaka, Y., Schwartz, S.E. (eds.) Diagrams 2016, pp. 289–292. LNCS 9781, Springer (2016)
45. Smessaert, H., Demey, L.: Duality patterns in 2-PCD fragments. South American Journal of Logic (2017)
46. Smessaert, H., Demey, L.: The unreasonable effectiveness of bitstrings in logical geometry. In: Béziau, J.Y., Basti, G. (eds.) The Square of Opposition: A Cornerstone of Thought, pp. 197–214. Springer (2017)
47. Westerståhl, D.: Classical vs. modern squares of opposition, and beyond. In: Béziau, J.Y., Payette, G. (eds.) The Square of Opposition. A General Framework for Cognition, pp. 195–229. Peter Lang (2012)
48. Yao, Y.: Duality in rough set theory based on the square of opposition. Fundamenta Informaticae 127, 49–64 (2013)